

GRAPHS WITH TWO TRIVIAL CRITICAL IDEALS

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ABSTRACT. The critical ideals of a graph are the determinantal ideals of the generalized Laplacian matrix associated to a graph. A basic property of the critical ideals of graphs asserts that the graphs with at most k trivial critical ideals, $\Gamma_{\leq k}$, are closed under induced subgraphs. In this article we find the set of minimal forbidden subgraphs for $\Gamma_{\leq 2}$, and we use this forbidden subgraphs to get a classification of the graphs in $\Gamma_{\leq 2}$. As a consequence we give a classification of the simple graphs whose critical group has two invariant factors equal to one. At the end of this article we give two infinite families of forbidden subgraphs.

1. INTRODUCTION

Given a connected graph $G = (V(G), E(G))$ and a set of indeterminates $X_G = \{x_u \mid u \in V(G)\}$, the *generalized Laplacian matrix* $L(G, X_G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where m_{uv} is the *multiplicity* of the edge uv , that is, the number of the edges between vertices u and v of G . For all $1 \leq i \leq n$, the *i-critical ideal* of G is the determinantal ideal given by

$$I_i(G, X_G) = \langle \{\det(m) \mid m \text{ is a square submatrix of } L(G, X_G) \text{ of size } i\} \rangle \subseteq \mathbb{Z}[X_G].$$

We say that a critical ideal is trivial when it is equal to $\langle 1 \rangle$.

Critical ideals are a generalization of the characteristic polynomials of the adjacency matrix and the Laplacian matrix associated to a graph. Also, critical ideals generalize the critical group of a graph as shown below: if $d_G(u)$ is the degree of a vertex u of G , then the *Laplacian matrix* of G , denoted by $L(G)$, is the evaluation of $L(G, X_G)$ on $x_u = d_G(u)$. Given a vertex s of G , the *reduced Laplacian matrix* of G , denoted by $L(G, s)$, is the matrix obtained from $L(G)$ by removing the row and column s . The *critical group* of a connected graph G , denoted by $K(G)$, is the cokernel of $L(G, s)$. That is,

$$K(G) = \mathbb{Z}^{\tilde{V}} / \text{Im } L(G, s),$$

where $\tilde{V} = V(G) \setminus s$. Therefore the critical group of a graph can be obtained from their critical ideals as shows [3, theorems 3.6 and 3.7]. The critical group have been studied intensively on several contexts over the last 30 years. However, a well understanding of the combinatorial and algebraic nature of the critical group still remains.

Let assume that G is a connected graph with n vertices. A classical result (see [6, section 3.7]) asserts that the reduced Laplacian matrix is equivalent to a integer diagonal matrix with entries d_1, d_2, \dots, d_{n-1}

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where $d_i > 0$ and $d_i \mid d_j$ if $i \leq j$. The integers d_1, \dots, d_{n-1} are unique and are called *invariant factors*. With this in mind, the critical group is described in terms of the invariant factors as follows [8, theorem 1.4]:

$$K(G) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \dots \oplus \mathbb{Z}_{d_{n-1}}.$$

Given an integer number k , let $f_k(G)$ be the number of invariant factors of the Laplacian matrix of G equal to k . Let $\mathcal{G}_i = \{G \mid G \text{ is a simple connected graph with } f_1(G) = i\}$. The study and characterization of \mathcal{G}_i is of great interest. In particular, some results and conjectures on the graphs with cyclic critical group can be found in [10, section 4] and [13, conjectures 4.3 and 4.4]. On the other hand, Dino Lorenzini, notice in [9] that \mathcal{G}_1 consists only of the complete graphs. More recently, Merino in [11] posed interest on the characterization of \mathcal{G}_2 and \mathcal{G}_3 . In this sense, few attempts have done. For instance, in [12] it was characterized the graphs in \mathcal{G}_2 whose third invariant factor is equal to n , $n-1$, $n-2$, or $n-3$. In [2] the characterizations of the graphs in \mathcal{G}_2 with a cut vertex, and the graphs in \mathcal{G}_2 with number of independent cycles equal to $n-2$ are given.

If $\Gamma_{\leq i}$ denotes the family of graphs with at most i trivial critical ideals, then it is not difficult to see that $\mathcal{G}_i \subseteq \Gamma_{\leq i}$ for all $i \geq 0$. At first glance, critical ideals behave better than critical ideals. For instance, by [3, proposition 3.3] we have that $\Gamma_{\leq i}$ is closed under induced subgraphs at difference of \mathcal{G}_i . This property will play a crucial role in order to get a characterization of $\Gamma_{\leq 2}$ on this paper. Also, if Γ_i is the family of graphs with exactly i trivial critical ideals, then we will shown on this paper that Γ_2 has a more simple description than \mathcal{G}_2 .

The main goals of this paper are three: to get a characterization of the graphs with at most two trivial critical ideals, to get a characterization of the graphs with two invariant factors equal to one, and to give two infinite families of forbidden subgraphs for $\Gamma_{\leq i}$.

This article is divided as follows: We begin by recalling some basic concepts on graph theory in section 2 and establishing some of basic properties of critical ideals in section 3. In section 4 we will characterize the graphs with at most two trivial critical ideals by finding their minimal set of forbidden graphs. As consequence, we will get the characterization of the graphs with two invariant factors equal to one. Finally, in section 5 we give two infinite families of forbidden graphs for $\Gamma_{\leq i}$.

2. BASIC DEFINITIONS

In this section, we give some basic definitions and notation of graph theory used in later sections. It should be pointed that we will consider the natural number as the the non-negative integers.

Given a graph $G = (V, E)$ and a subset U of V , the subgraph of G induced by U will be denoted by $G[U]$. If u is a vertex of G , let $N_G(u)$ be the set of *neighbors* of u in G . Here a *clique* of a graph G is a maximum complete subgraph, and its order is the *clique number* of G , denoted by $\omega(G)$. The *path* with n vertices is denoted by P_n , a *matching* with k edges by M_k , the *complete graph* with n vertices by K_n and the *trivial graph* of n vertices by T_n . The *cone* of a graph G is the graph obtained from G by adding a new vertex, called *apex*, which is adjacent to each vertex of G . The cone of a graph G is denoted by $c(G)$. Thus, the *star* S_k of $k+1$ vertices is equal to $c(T_k)$. Given two graphs G and H , their *union* is denoted by $G \cup H$, and their *disjoint union* by $G + H$. The *join* of G and H , denoted by $G \vee H$, is the graph obtained from $G + H$ when we add all the edges between vertices of G and H . For $m, n, o \geq 1$, let $K_{m,n,o}$ be the *complete tripartite graph*. You can consult [4] for any unexplained concept of graph theory.

Let $M \in M_n(\mathbb{Z})$ be a $n \times n$ matrix with entries on \mathbb{Z} , $\mathcal{I} = \{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$, and $\mathcal{J} = \{j_1, \dots, j_s\} \subseteq \{1, \dots, n\}$. The submatrix of M formed by rows i_1, \dots, i_r and columns j_1, \dots, j_s is

denoted by $M[\mathcal{I}; \mathcal{J}]$. If $|\mathcal{I}| = |\mathcal{J}| = r$, then $M[\mathcal{I}; \mathcal{J}]$ is called a r -square submatrix or a square submatrix of size r of M . A r -minor is the determinant of a r -square submatrix. The set of i -minors of a matrix M will be denoted by $\text{minors}_i(M)$. Finally, the identity matrix of size n is denoted by I_n and the all ones $m \times n$ matrix is denoted by $J_{m,n}$. We say that $M, N \in M_n(\mathbb{Z})$ are *equivalent*, denoted by $N \sim M$, if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$. Note that if $N \sim M$, then $K(M) = \mathbb{Z}^n / M^t \mathbb{Z}^n \cong \mathbb{Z}^n / N^t \mathbb{Z}^n = K(N)$.

3. GRAPHS WITH FEW TRIVIAL CRITICAL IDEALS

In this section, we will introduce the critical ideals of a graph and the set of graphs with k or less trivial critical ideals, denoted by $\Gamma_{\leq k}$. After that, we define the set of minimal forbidden graphs of $\Gamma_{\leq k}$. We finish this section with the classification of \mathcal{G}_1 , that we already know that they are the complete graphs.

Let G be a graph and $X_G = \{x_v \mid v \in V(G)\}$ be the set of indeterminates indexed by the vertices of G . For all $1 \leq i \leq n$, the i -critical ideal $I_i(G, X_G)$ is defined as the ideal of $\mathbb{Z}[X_G]$ given by

$$I_i(G, X_G) = \langle \{\det(m) \mid m \text{ is a square matrix of } L(G, X_G) \text{ of size } i\} \rangle.$$

By convention $I_i(G, X_G) = \langle 1 \rangle$ if $i < 1$, and $I_i(G, X_G) = \langle 0 \rangle$ if $i > n$. The *algebraic co-rank* of G , denoted by $\gamma(G)$, is the number of critical ideals of G equal to $\langle 1 \rangle$.

Definition 3.1. For all $k \in \mathbb{N}$, let $\Gamma_{\leq k} = \{G \mid G \text{ is a simple connected graph with } \gamma(G) \leq k\}$ and $\Gamma_{\geq k} = \{G \mid G \text{ is a simple connected graph with } \gamma(G) \geq k\}$.

Note that, $\Gamma_{\leq k}$ and $\Gamma_{\geq k+1}$ are disjoint sets and that a characterization of one of them leads to a characterization of the other one. Now, let us recall some basic properties about critical ideals, see [3] for details. It is known that if $i \leq j$, then $I_j(G, X_G) \subseteq I_i(G, X_G)$. Moreover, if H is an induced subgraph of G , then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$ and therefore $\gamma(H) \leq \gamma(G)$. This implies that $\Gamma_{\leq k}$ is closed under induced subgraphs, that is, if $G \in \Gamma_{\leq k}$ and H is an induced subgraph of G , then $H \in \Gamma_{\leq k}$.

Definition 3.2. Let $f_k(G)$ be the number of invariant factors of $K(G)$ that are equal to k and

$$\mathcal{G}_i = \{G \mid G \text{ is a simple connected graph with } f_1(G) = i\}.$$

Presumably $\Gamma_{\leq k}$ behaves better than \mathcal{G}_k . It is not difficult to see that unlike of $\Gamma_{\leq k}$, \mathcal{G}_k is not closed under induced subgraphs. For instance, consider $c(S_3)$, clearly it belongs to \mathcal{G}_2 , but S_3 belongs to \mathcal{G}_3 . Similarly, $K_6 \setminus \{2P_2\}$ belongs to \mathcal{G}_3 meanwhile $K_5 \setminus \{2P_2\}$ belongs to \mathcal{G}_2 . Moreover, if H is an induced subgraph of G , it is not always true that $K(H) \leq K(G)$. For example, $K(K_4) \cong \mathbb{Z}_4^2 \not\leq K(K_5) \cong \mathbb{Z}_5^3$. Finally, theorems 4.2 and 4.14 gives us additional evidence in the sense that $\Gamma_{\leq k}$ behaves better than \mathcal{G}_k . Moreover, theorem 3.6 of [3] implies that $\gamma(G) \leq f_1(G)$ for any graph and therefore $\mathcal{G}_k \subseteq \Gamma_{\leq k}$ for all $k \geq 0$.

A graph G is *forbidden* (or an *obstruction*) for $\Gamma_{\leq k}$ if and only if $\gamma(G) \geq k+1$. Let $\mathbf{Forb}(\Gamma_{\leq k})$ be the set of minimal (under induced subgraphs property) forbidden graphs for $\Gamma_{\leq k}$. Also, a graph G is called γ -critical if $\gamma(G \setminus v) < \gamma(G)$ for all $v \in V(G)$. That is, $G \in \mathbf{Forb}(\Gamma_{\leq k})$ if and only if G is γ -critical with $\gamma(G) = k+1$.

Given a family of graphs \mathcal{F} , a graph G is called \mathcal{F} -free if no induced subgraph of G is isomorphic to a member of \mathcal{F} . Thus, G belongs to $\Gamma_{\leq k}$ if and only if G is $\mathbf{Forb}(\Gamma_{\leq k})$ -free, or equivalently, G belongs to $\Gamma_{\geq k+1}$ if and only if G contains a graph of $\mathbf{Forb}(\Gamma_{\leq k})$ as an induced subgraph.

These ideas are useful to characterize $\Gamma_{\leq k}$. For instance, since $\gamma(P_2) = 1$ and no one of its induced subgraphs has $\gamma \geq 1$, then $P_2 \in \mathbf{Forb}(\Gamma_{\leq 0})$. Moreover, it is easy to see that T_1 is the only connected graph that is P_2 -free. Thus, since $I_1(T_1, \{x\}) \neq \langle 1 \rangle$, we get that $\mathbf{Forb}(\Gamma_{\leq 0}) = \{P_2\}$, and $\Gamma_{\leq 0}$ consists of the graph with one vertex. Also, it is not difficult to prove that $\mathcal{G}_0 = \Gamma_{\leq 0}$ and that the set of non-necessarily connected graphs with algebraic co-rank equal to zero consists only of the trivial graphs. In the next section we will use this kind of arguments in order to get $\mathbf{Forb}(\Gamma_{\leq k})$ and characterize $\Gamma_{\leq k}$ for k equal to 1 and 2. Finally, section 5 will be devoted to explore in general the set $\mathbf{Forb}(\Gamma_{\leq k})$.

Now, we obtain the characterization of $\Gamma_{\leq 1}$.

Theorem 3.3. *If G is a simple connected graph, then the following statements are equivalent:*

- (i) $G \in \Gamma_{\leq 1}$,
- (ii) G is P_3 -free,
- (iii) G is a complete graph.

Proof. (i) \Rightarrow (ii) Since $\gamma(P_3) = 2$, then clearly G must be P_3 -free.

(ii) \Rightarrow (iii) If G is not a complete graph, then it has two vertices not adjacent, say u and v . Let P be the smallest path between u and v . Thus, the length of P is greater or equal to 3. So, P_3 is an induced subgraph of P and hence of G . Therefore, G is a complete graph.

(iii) \Rightarrow (i) It is easy to see that for any non-trivial simple connected graph, its first critical ideal is trivial, meanwhile $I_1(K_1, \{x\}) = \langle x \rangle$. On the other hand, the 2-minors of a complete graphs are of the forms: $-1 + x_i x_j$ and $1 + x_i$. Since $-1 + x_i x_j \in \langle 1 + x_1, \dots, 1 + x_n \rangle$, then

$$(3.1) \quad I_2(K_n, X_{K_n}) = \begin{cases} \langle -1 + x_1 x_2 \rangle & \text{if } n = 2, \text{ and,} \\ \langle 1 + x_1, \dots, 1 + x_n \rangle & \text{if } n \geq 3. \end{cases}$$

Therefore $\gamma(K_n) \leq 1$. In fact, the set $\{1 + x_1, \dots, 1 + x_n\}$ is a reduced Gröbner basis for $I_2(K_n, X_{K_n})$, see [3, theorem 3.14]. \square

In light of theorem 3.3, the characterization of \mathcal{G}_1 is as follows: Clearly, $\mathcal{G}_1 \subseteq \Gamma_{\leq 1} \setminus \mathcal{G}_0$. Now, let $G \in \Gamma_{\leq 1} \setminus \{K_1\}$, that is, $G = K_n$ with $n \geq 2$ and $f_1(G) \geq 1$. It is easy to verify from equation 3.1 that the second invariant factor of $K(G)$ is equal to $I_2(K_n, X_{K_n})|_{\{x_v = n-1 \mid v \in K_n\}}$ which is different to 1.

Corollary 3.4. [9] *If G is a simple connected graph with $n \geq 2$ vertices, then $f_1(G) = 1$ if and only if G is a complete graph.*

A crucial fact in the proof of theorem 3.3 was that P_3 belongs to $\mathbf{Forb}(\Gamma_{\leq 1})$ and the fact that any other connected simple graph belonging to $\Gamma_{\geq 2}$ contains P_3 . This leads to the following corollary.

Corollary 3.5. $\mathbf{Forb}(\Gamma_{\leq 1}) = \{P_3\}$.

Next corollary give us the non-connected version of theorem 3.3.

Corollary 3.6. *If G is a simple non-necessary connected graph, then the following statements are equivalent:*

- (i) $\gamma(G) \leq 1$,
- (ii) G is $\{P_3, 2P_2\}$ -free,
- (iii) G is a disjoint union of a complete graph and a trivial graph.

Before to proceed with the proof of corollary 3.6 present a lemma that help us to calculate the critical ideal of a non connected graph. It may be useful to recall that the *product* of the ideals I and J of a commutative ring R , which we denote by IJ , is the ideal generated by all the products ab where $a \in I$ and $b \in J$.

Lemma 3.7. [3, Proposition 3.4] *If G and H are vertex disjoint graphs, then*

$$I_i(G + H, \{X_G, Y_H\}) = \langle \bigcup_{j=0}^i I_j(G, X_G) I_{i-j}(H, Y_H) \rangle \text{ for all } 1 \leq i \leq |V(G + H)|.$$

By this lemma we have that $\gamma(G + H) = \gamma(G) + \gamma(H)$ when G and H are vertex disjoint.

Proof of corollary 3.6. (i) \Rightarrow (ii) It follows since $\gamma(2P_2) = 2$ and $\gamma(P_3) = 2$.

(ii) \Rightarrow (iii) Let G_1, \dots, G_s be the connected components of G . Then by theorem 3.3 and lemma 3.7, G_i is a complete graph for all $1 \leq i \leq s$. Since $2P_2$ must not be an induced subgraph of G , then at most one of the G_i has order greater than 1.

(iii) \Rightarrow (i) If $G = K_n + T_m$, then it is not difficult to see that $I_1(T_m, Y_{T_m}) = \langle y_1, \dots, y_m \rangle$ and $I_2(T_m, Y_{T_m}) = \langle \prod_{i \neq j} y_i y_j \rangle$. Thus by lemma 3.7,

$$I_2(G, \{X_{K_n}, Y_{T_m}\}) = \langle I_2(K_n, X_{K_n}), I_1(K_n, X_{K_n}) I_1(T_m, Y_{T_m}), I_2(T_m, Y_{T_m}) \rangle \neq \langle 1 \rangle.$$

□

4. GRAPHS WITH ALGEBRAIC CO-RANK EQUAL TO TWO

The main goal of this section is to classify the simple graphs on $\Gamma_{\leq 2}$. After that, using the fact that $\mathcal{G}_2 \subseteq \Gamma_{\leq 2}$ we will classify the simple graphs whose critical group has two invariant factors equal to 1. As in the case of $\Gamma_{\leq 1}$, the characterization of $\Gamma_{\leq 2}$ relies heavily in the fact that $\Gamma_{\leq 2}$ is closed under induced subgraphs and the fact that we have a good guessing about $\mathbf{Forb}(\Gamma_{\leq 2})$. We begin with the introduction of a set of graphs in $\mathbf{Forb}(\Gamma_{\leq 2})$.

Proposition 4.1. *Let \mathcal{F}_2 be the set of graphs consisting of P_4 , $K_5 \setminus S_2$, $K_6 \setminus M_2$, cricket and dart; see figure 1. Then $\mathcal{F}_2 \subseteq \mathbf{Forb}(\Gamma_{\leq 2})$.*

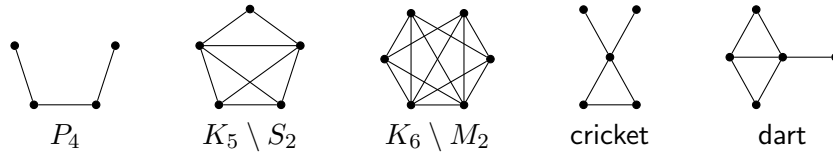


FIGURE 1. The set \mathcal{F}_2 of graphs.

Proof. It is not difficult to see that the generalized Laplacian matrix of the graphs on \mathcal{F}_2 are given by:

$$L(P_4) = \begin{bmatrix} x_1 & -1 & 0 & 0 \\ -1 & x_2 & -1 & 0 \\ 0 & -1 & x_3 & -1 \\ 0 & 0 & -1 & x_4 \end{bmatrix}, \quad L(K_5 \setminus S_2) = \begin{bmatrix} x_1 & 0 & -1 & -1 & 0 \\ 0 & x_2 & -1 & -1 & -1 \\ -1 & -1 & x_3 & -1 & -1 \\ -1 & -1 & -1 & x_4 & -1 \\ 0 & -1 & -1 & -1 & x_5 \end{bmatrix},$$

$$L(\text{cricket}) = \begin{bmatrix} x_1 & 0 & 0 & -1 & 0 \\ 0 & x_2 & -1 & -1 & 0 \\ 0 & -1 & x_3 & -1 & 0 \\ -1 & -1 & -1 & x_4 & -1 \\ 0 & 0 & 0 & -1 & x_5 \end{bmatrix}, \quad L(\text{dart}) = \begin{bmatrix} x_1 & -1 & 0 & -1 & 0 \\ -1 & x_2 & -1 & -1 & 0 \\ 0 & -1 & x_3 & -1 & 0 \\ -1 & -1 & -1 & x_4 & -1 \\ 0 & 0 & 0 & -1 & x_5 \end{bmatrix},$$

$$L(K_6 \setminus M_2) = \begin{bmatrix} x_1 & 0 & -1 & -1 & -1 & -1 \\ 0 & x_2 & -1 & -1 & -1 & -1 \\ -1 & -1 & x_3 & -1 & -1 & 0 \\ -1 & -1 & -1 & x_4 & -1 & -1 \\ -1 & -1 & -1 & -1 & x_5 & -1 \\ -1 & -1 & 0 & -1 & -1 & x_6 \end{bmatrix}.$$

In this matrices we marked with gray some 3×3 square submatrices whose determinant is equal to ± 1 . Then $\gamma(G) \geq 3$ for all $G \in \mathcal{F}_2$. Finally, using any algebraic system, for instance *Macaulay 2*, one can note that the graphs in \mathcal{F}_2 has algebraic co-rank equal to 3. Moreover, it can be checked that any of his induced subgraphs has algebraic co-rank less or equal to 2. \square

One of the main results of this article is the following:

Theorem 4.2. *Let G be a simple connected graph. Then, $G \in \Gamma_{\leq 2}$ if and only if G is an induced subgraph of $K_{m,n,o}$ or $T_n \vee (K_m + K_o)$.*

We divide the proof of theorem 4.2 in two steps. First we classify the connected graphs that are \mathcal{F}_2 -free. After that, we check that all these graphs have algebraic co-rank less or equal than two.

Theorem 4.3. *A simple connected graph is \mathcal{F}_2 -free if and only if it is an induced subgraph of $K_{m,n,o}$ or $T_n \vee (K_m + K_o)$.*

Proof. First, one implication is clear because $K_{m,n,o}$ and $T_n \vee (K_m + K_o)$ are \mathcal{F}_2 -free. The another part of the proof is divided in three cases: when $\omega(G) = 2$, $\omega(G) = 3$, and $\omega(G) \geq 4$.

The case when $\omega(G) = 2$ is very simple. Since $\omega(G) = 2$, there exist $a, b \in V(G)$ such that $ab \in E(G)$. Clearly, $N_G(a) \cap N_G(b) = \emptyset$. Moreover, if $x \in \{a, b\}$, then $uv \notin E(G)$ for all $u, v \in N_G(x)$. On the other hand, since G is P_4 -free, then $uv \in E(G)$ for all $u \in N_G(a)$ and $v \in N_G(b)$. Therefore G is the complete bipartite graph.

Now, assume that $\omega(G) = 3$. Let a, b and c be vertices of G that induce a complete graph. For all $X \subseteq \{a, b, c\}$ let $V_X = \{v \in V(G) \mid N_G(v) \cap \{a, b, c\} = X\}$. Clearly $V_{\{a,b,c\}} = \emptyset$ because $\omega(G) = 3$. In a similar way, if $X \subseteq \{a, b, c\}$ has size two, then set V_X induce a trivial graph. Also, since G is cricket-free, V_x induces a complete graph for all $x \in \{a, b, c\}$. Thus V_x has at most two vertices.

Now, given $U, V \in V(G)$, let $E(U, V) = \{uv \in E(G) \mid u \in U \text{ and } v \in V\}$. Let $x \neq y \in \{a, b, c\}$ and $z \in \{a, b, c\}$ such that $\{x, y, z\} = \{a, b, c\}$. Assume that V_x, V_y and $V_{\{x,y\}}$ are not empty. Let $u \in V_x$ and $v \in V_y$. If $uv \in E(G)$, then $\{u, v, y, z\}$ induced a P_4 . Therefore $E(V_x, V_y) = \emptyset$. In a similar way, since G is P_4 -free, we get $E(V_x, V_{\{x,y\}}) = \emptyset$.

Claim 4.4. *At least two of the sets V_a, V_b or V_c are empty. Furthermore, if $V_a \neq \emptyset$, then G is an induced subgraph of $T_l \vee (K_2 + K_2)$, where $l = |V_{\{b,c\}}| + 1$.*

Proof. First, assume that V_x and V_y are non empty. Let $u \in V_y, v \in V_x$. Since u and v are not adjacent, the vertices $\{u, x, y, v\}$ induces a P_4 . Therefore at least one of V_x or V_y is empty.

Without loss of generality we can assume that V_a is not empty. Since there is no edge between V_a and $V_{\{a,b\}}$, then $V_{\{a,b\}} = \emptyset$. Otherwise, if $u \in V_{\{a,b\}}$ and $v \in V_a$, then the vertices $\{u, v, a, b, c\}$ induces a **dart**. In a similar way $V_{\{a,c\}} = \emptyset$. On the other hand, if $V_{\{b,c\}}$ is not empty and there exist $u \in V_{\{b,c\}}$ and $v \in V_a$ such that $uv \notin E(G)$, then the vertices $\{u, b, a, v\}$ induces a P_4 . Therefore, either $E(V_a, V_{\{b,c\}}) = \{uv \mid u \in V_a \text{ and } v \in V_{\{b,c\}}\}$ or the set $V_{\{b,c\}}$ is empty. Finally, since V_a is a complete graph with at most two vertices, the result follows. \square

Now, we can assume that $V_x = \emptyset$ for all $x \in \{a, b, c\}$. Let $\{x, y, z\} = \{a, b, c\}$. If $uv \notin E(G)$ for some $u \in V_{\{x,y\}}$, $v \in V_{\{x,z\}}$, then $\{u, y, z, v\}$ induces a P_4 . Therefore $uv \in E(G)$ for all $u \in V_{\{x,y\}}$ and $v \in V_{\{x,z\}}$, and G is an induced subgraph of the complete tripartite graph.

We finish with case when $\omega(G) \geq 4$. Let $W = \{a, b, c, d\}$ be a complete subgraph of G of size four and let

$$V_i = \{v \in V(G) \setminus W \mid |N_G(v) \cap W| = i\} \text{ for all } i = 0, 1, 2, 3, 4.$$

Since G is $K_5 \setminus S_2$ -free, $V_2 = \emptyset$.

Claim 4.5. *The graph induced by V_1 is a complete graph.*

Proof. Let $u, u' \in V_1$ and suppose there is no edge between u and u' . Let $x, y \in W$ be the vertices adjacent to u and u' , respectively. If $x \neq y$, then $\{u, x, y, u'\}$ induces a P_4 ; a contradiction. On the other hand, if $x = y$, let $z \neq w \in W \setminus x$. Since u and u' are not adjacent to both z and w , then $\{x, z, w, u, u'\}$ induces a cricket; a contradiction. \square

Let $v, v' \in V_3$ and assume that are adjacent. Let $x, y \in W$ such that $x \notin N_G(v)$ and $y \notin N_G(v')$. If $x \neq y$, then $\{v, v'\} \cup W$ induces a $K_6 \setminus M_2$; a contradiction. On the other hand, if $x = y$, then $\{v, v'\} \cup W$ contains a $K_5 \setminus S_2$ as induced graph; a contradiction. Therefore V_3 induces a trivial graph.

Now, let $u \in V_1$, $v \in V_3$, $x, y \in W$ such that $xu \in E(G)$, $yv \notin E(G)$. Assume that $uv \notin E(G)$. Let $z \in W \setminus \{x, y\}$. If $x = y$, then $\{v, z, x, u\}$ induces a P_4 ; a contradiction. On the other hand, if $x \neq y$, then G must contains a dart as induced subgraph; a contradiction. Therefore $E(V_1, V_3)$ contains all the possible edges. Since $uv \in E(G)$, then $x = y$. Otherwise, if $x \neq y$, then $\{y, z, v, u\}$ induces a P_4 ; a contradiction. Therefore we can assume without loss of generality that $\{a\} = N_G(V_1) \cap W = (N_G(V_3) \cap W)^c$.

Now, let $w \in V_4$, $u \in V_1$, and $v \in V_3$. If $uw \in E(G)$, then $\{u, w, a, b, c\}$ induces a $K_5 \setminus S_2$. Therefore, $E(V_1, V_4) = \emptyset$. In a similar way, if $vw \notin E(G)$, then $\{v, a, w, b, c\}$ induces a $K_5 \setminus S_2$. Therefore, $E(V_3, V_4) = \{vw \mid v \in V_3 \text{ and } w \in V_4\}$.

Since G is $\{K_5 \setminus S_2, K_6 \setminus M_2\}$ -free, then it is not difficult to see that the graph induced by V_4 is $\{K_2 + T_1, C_4\}$ -free. Thus V_4 induces either a trivial graph, a complete graph, or a complete graph minus an edge. Moreover, if $ww' \notin E(G)$ for some $w \neq w' \in V_4$ and $v \in V_3$, then $\{w, w', a, v, b, c\}$ induces a $K_6 \setminus M_2$. Thus, if $V_3 \neq \emptyset$, then V_4 induces a complete graph.

Clearly, if $V_1, V_3, V_4 = \emptyset$, then G is a complete graph.

Claim 4.6. *If $V_1, V_3 = \emptyset$ and $V_4 \neq \emptyset$, then G is an induced subgraph of $T_1 \vee (K_m + K_n)$ for some $m, n \in \mathbb{N}$.*

Proof. If $|V_0| = |V_4| = 1$, then the result is clear. Therefore we can assume that $|V_4| \geq 2$ or $|V_0| \geq 2$. Moreover we need to consider three cases for V_4 , when it induces a trivial graph, a complete graph, or a complete graph minus an edge. Assume that V_4 induces a trivial graph. If $|V_4| \geq 2$, let $o \in V_0$ and $w, w' \in V_4$. If $ow \in E(G)$ and $ow' \notin E(G)$, then $\{o, w, w', a\}$ induces a P_4 ; a contradiction. Thus, either $E(o, V_4) = \{ow \mid w \in V_4\}$ or $E(o, V_4)$ is empty. Therefore, since G is connected, we get the result when $|V_0| = 1$.

Now, assume that $|V_0| \geq 2$. Since G is connected, there exist $o \in V_0$ such that $ow \in E(G)$ for some $w \in V_4$. Let $o' \in V_0$ such that $E(o', V_4)$ is empty. Since G is connected, there exist a path from o' to o . Let P be a minimum path between o' and o . In this case, $\{V(P), w, a\}$ induces a path with more than four vertices; a contradiction. Therefore, $E(V_0, V_4) = \{ow \mid o \in V_0 \text{ and } w \in V_4\}$. Moreover, since G is $K_6 \setminus M_2$ -free, then V_0 induces a trivial graph and we get the result.

Now, assume that V_4 induces a complete graph. Since G is $K_5 \setminus S_2$ -free, o is adjacent to at most one vertex in V_4 . Moreover, all the vertices in V_0 are adjacent to a unique vertex in V_4 . Otherwise, let $o, o' \in V_0$ and $w, w' \in V_4$ such that $ow, o'w' \in E(G)$ and $ow', o'w \notin E(G)$. If $oo' \in E(G)$, then $\{a, w, o, o'\}$ induces a P_4 ; a contradiction. Also, if $oo' \notin E(G)$ and $ww' \in E(G)$, then $\{w, w', o, o'\}$ induces a P_4 ; a contradiction. Let $w \in V_4$ such that all the vertices in V_0 are adjacent to w . Then V_0 induces a complete graph. Otherwise, $\{a, b, w, o, o'\}$ induces a cricket; a contradiction. Therefore G is an induced subgraph of $T_1 \vee (K_m + K_n)$ for some $m, n \in \mathbb{N}$.

Finally, when V_4 induces a complete graph minus an edge, following similar arguments to those given in the case when V_4 induces a complete graph we get that G is an induced subgraph of $T_2 \vee (K_m + K_n)$ for some $m, n \in \mathbb{N}$. \square

Therefore we can assume that $V_1 \cup V_3 \neq \emptyset$. Let $u \in V_1 \cup V_3$, $o \in V_0$, and $x \neq y \in W$ such that $x \notin N_G(u)$ and $y \in N_G(u)$. If $uo \in E(G)$, then $\{x, y, u, o\}$ induces a P_4 ; a contradiction. Thus, there are no edges between V_0 and $V_1 \cup V_3$. Moreover, let $w \in V_4$. If $ow \in E(G)$, then $\{a, b, u, w, o\}$ induces a dart when $u \in V_3$ and $\{u, a, w, o\}$ induces a P_4 when $u \in V_1$. Therefore there are no edges between V_0 and V_4 . Since G is connected, $V_0 = \emptyset$ and therefore G is an induced subgraph of $T_n \vee (K_m + K_o)$. \square

To finish the proof of theorem 4.2 we need to prove that the third critical ideal of the graphs $K_{m,n,o}$ and $T_n \vee (K_m + K_o)$ is not trivial. If $m + n + o \leq 2$, then the third critical ideal is equal to zero. Also, if $m + n + o = 3$, then the third critical ideal is equal to the determinant of the generalized Laplacian matrix. Moreover, [3, theorem 3.16] proves that the algebraic co-rank of the complete graph is equal to 1.

Theorem 4.7. *If $K_{m,n,o}$ is connected with $m \geq n \geq o$ and $m + n + o \geq 4$, then*

$$(4.1) \quad I_3(K_{m,n,o}, \{X, Y, Z\}) = \begin{cases} \langle 2, \bigcup_{i=1}^m x_i, \bigcup_{i=1}^n y_i, \bigcup_{i=1}^o z_i \rangle & \text{if } m, n, o \geq 2, \\ \langle \bigcup_{i=1}^m x_i, \bigcup_{i=1}^n y_i, z_1 + 2 \rangle & \text{if } m \geq 2, n \geq 2, o = 1, \\ \langle \bigcup_{i=1}^m x_i, y_1 + z_1 + 2 \rangle & \text{if } m \geq 3, n = 1, o = 1, \\ \langle x_1 x_2 + x_1 + x_2, x_1 z_1 + x_1, x_2 z_1 + x_2, y_1 + z_1 + 2 \rangle & \text{if } m = 2, n = 1, o = 1, \\ \langle \bigcup_{i=1}^m x_i, \bigcup_{i=1}^n y_i \rangle & \text{if } m \geq 3, n \geq 3, o = 0, \\ \langle \bigcup_{i=1}^m x_i, y_1 + y_2 \rangle & \text{if } m \geq 3, n = 2, o = 0, \\ \langle x_2 y_2, x_1 + x_2, y_1 + y_2 \rangle & \text{if } m = 2, n = 2, o = 0, \\ \langle \bigcup_{i=1}^m x_i \rangle & \text{if } m \geq 3, n = 1, o = 0. \end{cases}$$

Theorem 4.8. *If $T_n \vee (K_m + K_o)$ is connected with $m \geq o$, $m + n + o \geq 4$, and such that $T_n \vee (K_m + K_o)$ is not the complete graph or the complete bipartite graph, then*

$$(4.2) \quad I_3(T_n \vee (K_m + K_o), \{X, Y, Z\}) = \begin{cases} \langle 2, \bigcup_{i=1}^m (x_i + 1), \bigcup_{i=1}^n y_i, \bigcup_{i=1}^o (z_i + 1) \rangle & \text{if } m, n, o \geq 2, \\ \langle \bigcup_{i=1}^m (x_i + 1), y_1 + 2, \bigcup_{i=1}^o (z_i + 1) \rangle & \text{if } m \geq 2, n = 1, o \geq 2, \\ \langle \bigcup_{i=1}^m (x_i + 1), \bigcup_{i=1}^n y_i, z_1 - 1 \rangle & \text{if } m \geq 2, n \geq 2, o = 1, \\ \langle x_1 + z_1, \bigcup_{i=1}^n y_i \rangle & \text{if } m = 1, n \geq 3, o = 1, \\ \langle x_1 + z_1, y_1 + y_2, y_2 z_1 \rangle & \text{if } m = 1, n = 2, o = 1, \\ \langle \bigcup_{i=1}^m (x_i + 1), z_1 y_1 + z_1 - 1 \rangle & \text{if } m \geq 2, n = 1, o = 1, \\ \langle \bigcup_{i=1}^m (x_i + 1), \bigcup_{i=1}^n y_i \rangle & \text{if } m \geq 3, n \geq 3, o = 0, \\ \langle x_1 + x_2 + 2, \bigcup_{i=1}^n y_i \rangle & \text{if } m = 2, n \geq 3, o = 0, \\ \langle \bigcup_{i=1}^m (x_i + 1), y_1 y_2 + y_1 + y_2 \rangle & \text{if } m \geq 3, n = 2, o = 0, \\ \langle x_1 + x_2 + 2, x_2 y_1 + y_1, x_2 y_2 + y_2, y_1 y_2 + y_1 + y_2 \rangle & \text{if } m = 2, n = 2, o = 0, \end{cases}$$

The proofs of theorems 4.7 and 4.8 relies on the description of the 3-minors of the generalized Laplacian matrices of $K_{m,n,o}$ and $T_n \vee (K_m + K_o)$.

Proof of theorem 4.7. In order to simplify the arguments in the proof we separate it in two parts. We begin by finding the 3-minors of the generalized Laplacian matrix of the complete bipartite graph and using it to calculate their third critical ideal. An after that, we do the same for the general case of the complete tripartite graph.

Lemma 4.9. *For $m, n \geq 1$, let $L_{m,n}$ be the generalized Laplacian matrix of the complete bipartite graph $K_{m,n}$. That is,*

$$L_{m,n} = L(K_{m,n}, \{X_{T_m}, Y_{T_n}\}) = \begin{bmatrix} L(T_m, X_{T_m}) & -J_{m,n} \\ -J_{n,m} & L(T_n, Y_{T_n}) \end{bmatrix}.$$

Then 3-minors (with positive leading coefficient) of $L_{m,n}$ are the following:

- $y_{j_1}, y_{j_1}y_{j_2}$, and $y_{j_1}y_{j_2}y_{j_3}$ when $n \geq 3$,
- $y_{j_1}y_{j_2}x_{i_1} - y_{j_1} - y_{j_2}$ when $n \geq 2$,
- $x_{i_1} + x_{i_2}, y_{j_1} + y_{j_2}$ and $x_{i_1}y_{j_1}$ when $m \geq 2$ and $n \geq 2$,
- $x_{i_1}, x_{i_1}x_{i_2}$, and $x_{i_1}x_{i_2}x_{i_3}$ when $m \geq 3$,
- $x_{i_1}x_{i_2}y_{j_1} - x_{i_1} - x_{i_2}$ when $m \geq 2$,

where $1 \leq i_1 < i_2 < i_3 \leq n$ and $1 \leq j_1 < j_2 < j_3 \leq n$.

Proof. Before to proceed with the proof we establish some notation corresponding to row and column indices. Let $\mathcal{I} = \{i_1, i_2, i_3\}$ such that $1 \leq i_1 < i_2 < i_3 \leq m + n$, and $\mathcal{J} = \{j_1, j_2, j_3\}$ such that $1 \leq j_1 < j_2 < j_3 \leq m + n$. Let $\mathcal{I}_1 = \mathcal{I} \cap [m]$, $\mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1$, $\mathcal{J}_1 = \mathcal{J} \cap [m]$, and $\mathcal{J}_2 = \mathcal{J} \setminus \mathcal{J}_1$. Also in the following $i'_t = i_t - m$ and $j'_t = j_t - m$, for all $1 \leq t \leq 3$.

In order to find all the 3-minors of $L_{m,n}$ we need to calculate the determinants of all non-singular matrices of the form $L_{m,n}[\mathcal{I}, \mathcal{J}]$. Since the generalized Laplacian matrix is symmetric, we can assume without loss of generalization that $|\mathcal{I}_2| \leq |\mathcal{J}_2|$. Let $L = L_{m,n}[\mathcal{I}, \mathcal{J}]$ be non-singular. First, consider the case when \mathcal{I}_2 is empty. Since the determinant of L is equal to zero when $|\mathcal{J}_2| \geq 2$, only remains to consider the cases when $|\mathcal{J}_2| = 0$ or $|\mathcal{J}_2| = 1$. If $|\mathcal{J}_2| = 0$, then $m \geq 3$, L is a submatrix of $L(T_m, X_{T_m})$, and the determinant of L is equal to $x_{i_1}x_{i_2}x_{i_3}$. In a similar way, if $|\mathcal{J}_2| = 1$, then $m \geq 3$, $n \geq 1$, and L is equal to (up to row permutation)

$$\begin{bmatrix} x_{j_1} & 0 & -1 \\ 0 & x_{j_2} & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

whose determinant is equal to $-x_{j_1}x_{j_2}$.

Now, consider the case when $|\mathcal{I}_2| = 1$. In a similar way, L has determinant different from zero when $|\mathcal{J}_2| = 1$ or $|\mathcal{J}_2| = 2$. If $|\mathcal{J}_2| = 1$, then there are essentially only four 3×3 non-singular submatrices of $L_{m,n}$:

$$\begin{bmatrix} x_{i_1} & 0 & -1 \\ 0 & A & -1 \\ -1 & -1 & B \end{bmatrix},$$

where A is equal to 0 (when $m \geq 3$) and x_{i_2} , and B is equal to 0 (when $n \geq 2$) and $y_{i'_3}$. Clearly $\det(L) = ABx_{i_1} - A - x_{i_1}$. Thus we have the following minors: $x_{i_1}x_{i_2}y_{i'_3} - x_{i_1} - x_{i_2}$, $-x_{i_1} - x_{i_2}$, $-x_{i_1}$. If $|\mathcal{J}_2| = 2$, then $m \geq 2$, $n \geq 2$, and L has determinant equal to

$$\det \begin{bmatrix} x_{j_1} & -1 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & y_{i'_3} \end{bmatrix} = -x_{j_1}y_{i'_3}.$$

When $|\mathcal{I}_2| = 2$ we have two cases, when either $|\mathcal{J}_2| = 2$ or $|\mathcal{J}_2| = 3$. If $|\mathcal{J}_2| = 2$, then L is equal to:

$$\begin{bmatrix} A & -1 & -1 \\ -1 & y_{i'_2} & 0 \\ -1 & 0 & B \end{bmatrix}$$

where A is equal to 0 (when $m \geq 2$) or x_{i_1} and B is equal to 0 (when $n \geq 3$) or $y_{i'_3}$. It is easy to see that $\det(L) = AB y_{i'_2} - A - y_{i'_2}$. Thus we have the following minors: $x_{i_1} y_{i'_2} y_{i'_3} - y_{i'_2} - y_{i'_3}$, $-y_{i'_2} - y_{i'_3}$, $-y_{i'_2}$. If $|\mathcal{J}_2| = 3$, then $m \geq 1$, $n \geq 3$ and there are only one non-singular matrix whose determinant is equal to

$$\det \begin{bmatrix} -1 & -1 & -1 \\ y_{i'_2} & 0 & 0 \\ 0 & y_{i'_3} & 0 \end{bmatrix} = -y_{i'_2} y_{i'_3}.$$

Finally, if $|\mathcal{I}_2| = 3$, then $n \geq 3$, L is a submatrix of $L(T_m, Y_{T_m})$, and therefore its determinant is equal to $y_{i'_1} y_{i'_2} y_{i'_3}$. \square

We can use lemma 4.9 to get the third critical ideal of the complete bipartite graph. For instance, it is not difficult to see that $I_3(K_{m,n}, \{X, Y\}) = \langle \bigcup_{i=1}^m x_i, \bigcup_{i=1}^n y_i \rangle$ when $m \geq 3$ and $n \geq 3$. In a similar way, since $x_{i_1} + x_{i_2}, x_{i_1} y_{j_1}, y_{j_1} y_{j_2} x_{i_1} - y_{j_1} - y_{j_2}, x_{i_1} x_{i_2}, x_{i_1} x_{i_2} x_{i_3} \in \langle \bigcup_{i=1}^m x_i, y_1 + y_2 \rangle$, $I_3(K_{m,n}, \{X, Y\}) = \langle \bigcup_{i=1}^m x_i, y_1 + y_2 \rangle$ when $m \geq 3$ and $n = 2$. The other cases follow in a similar way.

Therefore in order to calculate the third critical ideal of the complete tripartite graph we need to calculate their 3-minors as below.

Theorem 4.10. *For $m, n, o \geq 1$, let $L_{m,n,o}$ be the generalized Laplacian matrix of the tripartite complete graph $K_{m,n,o}$. That is,*

$$L_{m,n,o} = L(K_{m,n,o}, \{X_{T_m}, Y_{T_n}, Z_{T_o}\}) = \begin{bmatrix} L(T_m, X_{T_m}) & -J_{m,n} & -J_{m,o} \\ -J_{n,m} & L(T_n, Y_{T_n}) & -J_{n,o} \\ -J_{o,m} & -J_{o,n} & L(T_o, Z_{T_o}) \end{bmatrix}.$$

Then the 3-minors (with positive leading coefficient) of $L_{m,n,o}$ are the following:

- $x_{i_1}, x_{i_1} x_{i_2}$, and $x_{i_1} x_{i_2} x_{i_3}$ when $m \geq 3$,
- $y_{j_1}, y_{j_1} y_{j_2}$, and $y_{j_1} y_{j_2} y_{j_3}$ when $n \geq 3$,
- $z_{k_1}, z_{k_1} z_{k_2}$, and $z_{k_1} z_{k_2} z_{k_3}$ when $o \geq 3$,
- $x_{i_1}, y_{j_1}, x_{i_1} + 2, y_{j_1} + 2, x_{i_1} + x_{i_2}, y_{j_1} + y_{j_2}$, and $x_{i_1} y_{j_1}$ when $m \geq 2$ and $n \geq 2$,
- $x_{i_1}, z_{k_1}, x_{i_1} + 2, z_{k_1} + 2, x_{i_1} + x_{i_2}, z_{k_1} + z_{k_2}$, and $x_{i_1} z_{k_1}$ when $m \geq 2$ and $o \geq 2$,
- $y_{j_1}, z_{k_1}, y_{j_1} + 2, z_{k_1} + 2, y_{j_1} + y_{j_2}, z_{k_1} + z_{k_2}$, and $y_{j_1} z_{k_1}$, when $n \geq 2$ and $o \geq 2$,
- $y_{j_1} + z_{k_1} + 2, x_{i_1}(y_{j_1} + 1), x_{i_1}(z_{k_1} + 1), x_{i_1} x_{i_2} + x_{i_1} + x_{i_2}, x_{i_1} x_{i_2} y_{j_1} - x_{i_1} - x_{i_2}$, and $x_{i_1} x_{i_2} z_{k_1} - x_{i_1} - x_{i_2}$ when $m \geq 2$,
- $x_{i_1} + z_{k_1} + 2, y_{j_1}(x_{i_1} + 1), y_{j_1}(z_{k_1} + 1), y_{j_1} y_{j_2} + y_{j_1} + y_{j_2}, y_{j_1} y_{j_2} x_{i_1} - y_{j_1} - y_{j_2}$, and $y_{j_1} y_{j_2} z_{k_1} - y_{j_1} - y_{j_2}$ when $n \geq 2$,
- $x_{i_1} + y_{j_1} + 2, z_{k_1}(x_{i_1} + 1), z_{k_1}(y_{j_1} + 1), z_{k_1} z_{k_2} + z_{k_1} + z_{k_2}, z_{k_1} z_{k_2} x_{i_1} - z_{k_1} - z_{k_2}$, and $z_{k_1} z_{k_2} y_{j_1} - z_{k_1} - z_{k_2}$ when $o \geq 2$,

where $1 \leq i_1 < i_2 < i_3 \leq m$, $1 \leq j_1 < j_2 < j_3 \leq n$, and $1 \leq k_1 < k_2 < k_3 \leq o$.

Proof. We will follow a similar proof to the proof given for lemma 4.9. Let $\mathcal{I} = \{i_1, i_2, i_3\}$ with $1 \leq i_1 < i_2 < i_3 \leq m + n + o$ and $\mathcal{J} = \{j_1, j_2, j_3\}$ with $1 \leq j_1 < j_2 < j_3 \leq m + n + o$. Moreover, let $\mathcal{I}_1 = \mathcal{I} \cap [m]$, $\mathcal{I}_2 = \mathcal{I} \cap \{m + 1, \dots, m + n\}$, $\mathcal{I}_3 = \mathcal{I} \cap \{m + n + 1, \dots, m + n + o\}$, $\mathcal{J}_1 = \mathcal{J} \cap [m]$, $\mathcal{J}_2 = \mathcal{J} \cap \{m + 1, \dots, m + n\}$, $\mathcal{J}_3 = \mathcal{J} \cap \{m + n + 1, \dots, m + n + o\}$. Also, in the following $i'_t = i_t - m$, $i''_t = i_t - m - n$, $j'_t = j_t - m$ and $j''_t = j_t - m - n$, for $t \in [3]$.

Let $L = L_{m,n,o}[\mathcal{I}; \mathcal{J}]$. First, in the same way that in the proof of lemma 4.9 we can assume that L is non-singular. Several of the 3-minor of $L_{m,n,o}$ can be calculated using lemma 4.9. For instance, if $\mathcal{I}_i = \mathcal{J}_i = \emptyset$ for some $i = 1, 2, 3$, then L is a submatrix of $L(K_{m,n}, \{X_{T_m}, Y_{T_n}\})$ and the corresponding

3-minor can be calculated using lemma 4.9. Therefore we can assume that, if $\mathcal{I}_i = \emptyset$, then $\mathcal{J}_i \neq \emptyset$ for all $i = 1, 2, 3$. Moreover, if $\mathcal{I}_i = \emptyset$, then $|\mathcal{J}_i| = 1$ for all $i = 1, 2, 3$. Because otherwise either L will have two identical columns; a contradiction to the fact that L is non-singular. In a similar way, if $\mathcal{J}_i = \emptyset$, then $|\mathcal{I}_i| = 1$ for all $i = 1, 2, 3$. If $|\mathcal{I}_i| = 3$ for some $i = 1, 2, 3$, then L is a submatrix of the generalized Laplacian matrix of a complete bipartite graph. Therefore we can assume that $|\mathcal{I}_i| \leq 2$ and $|\mathcal{J}_i| \leq 2$ for all $i = 1, 2, 3$.

The first case that we need to consider is when $\mathcal{I}_i \neq \emptyset \neq \mathcal{J}_i$ for all $1 \leq i \leq 3$, that is, $|\mathcal{I}_i| = |\mathcal{J}_i| = 1$ for all $1 \leq i \leq 3$. In this case we have that

$$L = \begin{bmatrix} A & -1 & -1 \\ -1 & B & -1 \\ -1 & -1 & C \end{bmatrix},$$

where A is equal to 0 (when $m \geq 2$) or x_{i_1} , B is equal to 0 (when $n \geq 2$) or y_{i_2} , and C is equal to 0 (when $n \geq 2$) or z_{i_3} . Since $\det L = ABC - A - B - C - 2$ we get eight of the 3-minors of $L_{m,n,o}$. Since $|\mathcal{I}_i| \leq 2$ ($|\mathcal{J}_i| \leq 2$) for all $i = 1, 2, 3$, then there are no two \mathcal{I} 's (\mathcal{J} 's) empty. Therefore only remains the cases: when only one of the \mathcal{I} 's is empty and the case when one of the \mathcal{I} 's is empty and one of the \mathcal{J} 's is empty.

Consider the case when only one of the sets \mathcal{I} 's is empty, that is, $|\mathcal{J}_i| = 1$ for all $i = 1, 2, 3$. Assume that $\mathcal{I}_3 = \emptyset$. Then we need to consider the following two matrices (when $|\mathcal{I}_1| = 1$ and $|\mathcal{I}_2| = 2$):

$$L_1 = \begin{bmatrix} A & -1 & -1 \\ -1 & 0 & -1 \\ -1 & B & -1 \end{bmatrix} \quad \text{and} \quad L_2 = \begin{bmatrix} A & -1 & -1 \\ 0 & -1 & -1 \\ -1 & B & -1 \end{bmatrix},$$

where A is equal to 0 (when $m \geq 2$ and $m \geq 3$, respectively) or x_{i_1} and B is equal to 0 (when $n \geq 3$ and $n \geq 2$, respectively) or y_{i_2} . It is not difficult to see that $\det(L_1) = AB - B$ and $\det(L_2) = AB - A$. Thus, we get the minors $x_{i_1}y_{i_2} - y_{i_2}$ (when $n \geq 2$), $x_{i_1}y_{i_2} - x_{i_1}$ (when $m \geq 2$), $-y_{i_2}$ and $-x_{i_1}$ (when $m \geq 2$ and $n \geq 2$). We get similar 3-minors when $\mathcal{I}_2 = \emptyset$ or $\mathcal{I}_1 = \emptyset$.

Finally, consider the case when one of the \mathcal{I} 's is empty and one of the \mathcal{J} 's is empty. Assume that $\mathcal{I}_3 = \emptyset$ and $\mathcal{J}_2 = \emptyset$. Then $|\mathcal{I}_2| = 1$ and L is equal to:

$$\begin{bmatrix} A & 0 & -1 \\ 0 & A' & -1 \\ -1 & -1 & -1 \end{bmatrix},$$

where A is equal to 0 or x_{i_1} and A' is equal to 0 or x_{i_2} . Clearly $\det L = -AA' - A - A'$. Thus we get the 3-minors $x_{i_1}x_{i_2} + x_{i_1} + x_{i_2}$ (when $m \geq 2$) and x_{i_1} and x_{i_2} (when $m \geq 3$). Similarly when $\mathcal{J}_1 = \emptyset$ and the other cases. \square

The rest it follows by similar arguments to those in the case of the bipartite complete graph. \square

Proof of theorem 4.8. Following the proof of theorem 4.7 we need to find the 3-minors of the generalized Laplacian matrix of $T_n \vee (K_m + K_o)$. We begin with $K_m \vee T_n$ and after that we do the same for $T_n \vee (K_m + K_o)$. We omit the proofs of lemma 4.11 and theorem 4.12 because are rutinary and both follows by using similar arguments to those in lemma 4.9 and in theorem 4.10, respectively.

Lemma 4.11. *For $m, n \geq 1$, let $L'_{m,n}$ be the generalized Laplacian matrix of $K_m \vee T_n$. That is,*

$$L'_{m,n} = L(K_m \vee T_n, \{X_{K_n}, Y_{T_m}\}) = \begin{bmatrix} L(K_m, X_{K_m}) & -J_{m,n} \\ -J_{n,m} & L(T_n, Y_{T_n}) \end{bmatrix}.$$

Then the 3-minors (with positive leading coefficient) of $L'_{m,n}$ are the following:

- $y_{j_1}, y_{j_1}y_{j_2}$, and $y_{j_1}y_{j_2}y_{j_3}$ when $n \geq 3$,
- $(x_{i_1}+1)(x_{i_2}+1), (x_{i_1}+1)(y_{i_1}+1)$, and $x_{i_1}x_{i_2}x_{i_3}-x_{i_1}-x_{i_2}-x_{i_3}-2$ when $m \geq 3$,
- $x_{i_1}y_{j_1}y_{j_2}-y_{j_1}-y_{j_2}$ when $n \geq 2$,
- $x_{i_1}x_{i_2}y_{j_1}-x_{i_1}-x_{i_2}-y_{j_1}-2$ when $m \geq 2$,
- y_{j_1} when $m \geq 2$ and $n \geq 3$,
- $x_{i_1}+1$ when $m \geq 3$ and $n \geq 2$,
- $x_{i_1}+x_{i_2}+2, x_{i_1}+y_{j_1}, x_{i_1}y_{j_1}y_{j_2}$ and $y_{j_1}y_{j_2}+y_{j_1}+y_{j_2}$ when $m \geq 2$ and $n \geq 2$,

where $1 \leq i_1 < i_2 < i_3 \leq m$ and $1 \leq j_1 < j_2 < j_3 \leq n$.

Theorem 4.12. For $m, n, o \geq 1$, let $L'_{m,n,o}$ be the the generalized Laplacian matrix of $T_n \vee (K_m + K_o)$. That is,

$$L'_{m,n,o} = L(T_n \vee (K_m + K_o), \{X_{K_m}, Y_{T_n}, Z_{K_o}\}) = \begin{bmatrix} L(K_m, X_{K_m}) & -J_{m,n} & \mathbf{0}_{m,o} \\ -J_{n,m} & L(T_n, Y_{T_n}) & -J_{n,o} \\ \mathbf{0}_{o,m} & -J_{o,n} & L(K_o, Z_{K_o}) \end{bmatrix}.$$

Then the 3-minors (with positive leading coefficient) of $L'_{m,n,o}$ are the following:

- $x_{i_1}+1$ when $m \geq 3$ and $(o \geq 2$ or $n \geq 2)$,
- y_{j_1} when $n \geq 3$ and $(m \geq 2$ or $o \geq 2)$,
- $y_{j_1}, y_{j_1}y_{j_3}$, and $y_{j_1}y_{j_2}y_{j_3}$ when $n \geq 3$,
- $x_{i_1}y_{j_1}z_{k_1}-x_{i_1}-z_{k_1}$,
- $x_{i_1}+1, z_{k_1}(x_{i_1}+1), (x_{i_1}+1)(x_{i_2}+1), (x_{i_1}+1)(y_{j_1}+1)$, and $x_{i_1}x_{i_2}x_{i_3}-x_{i_1}-x_{i_2}-x_{i_3}-2$, when $m \geq 3$,
- $z_{k_1}+1, x_{i_1}(z_{k_1}+1), (z_{k_1}+1)(z_{k_2}+1), (z_{k_1}+1)(y_{j_1}+1)$, and $z_{k_1}z_{k_2}z_{k_3}-z_{k_1}-z_{k_2}-z_{k_3}-2$ when $o \geq 3$,
- $x_{i_1}+z_{k_1}, y_{j_1}+y_{j_2}, x_{i_1}y_{j_1}, y_{j_1}z_{k_1}, x_{i_1}y_{j_1}y_{j_2}-y_{j_1}-y_{j_2}$, and $y_{j_1}y_{j_2}z_{k_1}-y_{j_1}-y_{j_2}$ when $n \geq 2$,
- $x_{i_1}+1, x_{i_1}x_{i_2}-1, y_{j_1}z_{k_1}+z_{k_1}-1, z_{k_1}(x_{i_1}+1), x_{i_1}x_{i_2}z_{k_1}-z_{k_1}$, and $x_{i_1}x_{i_2}y_{j_1}-x_{i_1}-x_{i_2}-y_{j_1}-2$, when $m \geq 2$,
- $z_{k_1}+1, x_{i_1}(z_{k_1}z_{k_2}-1), z_{k_1}z_{k_2}-1, x_{i_1}y_{j_1}+x_{i_1}-1, x_{i_1}(z_{k_1}+1)$, and $z_{k_1}z_{k_2}y_{j_1}-z_{k_1}-z_{k_2}-y_{j_1}-2$, when $o \geq 2$,
- $x_{i_1}+1, y_{j_1}+2, z_{k_1}+1, x_{i_1}x_{i_2}-1$, and $z_{k_1}z_{k_2}-1$ when $m \geq 2$ and $o \geq 2$,
- $x_{i_1}+1, y_{j_1}, z_{k_1}-1, x_{i_1}+y_{j_1}, x_{i_1}+x_{i_2}+2, y_{j_1}(x_{i_1}+1)$, and $y_{j_1}y_{j_2}+y_{j_1}+y_{j_2}$ when $m \geq 2$ and $n \geq 2$,
- $x_{i_1}-1, y_{j_1}, z_{k_1}+1, z_{k_1}+y_{j_1}, z_{k_1}+z_{k_2}+2, y_{j_1}(z_{k_1}+1)$, and $y_{j_1}y_{j_2}+y_{j_1}+y_{j_2}$, when $n \geq 2$ and $o \geq 2$,

where $1 \leq i_1 < i_2 < i_3 \leq m$, $1 \leq j_1 < j_2 < j_3 \leq n$, and $1 \leq k_1 < k_2 < k_3 \leq o$. \square

Theorems 4.7 and 4.8 implies that $\text{Forb}(\Gamma_{\leq 2}) = \mathcal{F}_2$. Now, we present the non-connected version of theorem 4.2.

Corollary 4.13. A simple graph has algebraic co-rank equal to two if and only if is the disjoint union of a trivial graph with one of the following graphs:

- $K_{m,n,o}$, where $m \geq 2, n+o \geq 1$,
- $T_n \vee (K_m + K_o)$, where $m, o \geq 2, m, n, o \geq 1$, or $n \geq 2$ and $m+o \geq 1$.

Proof. It is not difficult to see that in the non-connected case we need to add the graphs $P_3 + P_2$ and $3P_2$ to the set of forbidden graphs. The rest follows directly from theorem 4.2. \square

We finish this section with the classification of the graphs having critical group with 2 invariant factors equal to one.

Theorem 4.14. The critical group of a connected simple graph has exactly two invariant factor equal to 1 if and only if is one of the following graphs:

- $K_{m,n,o}$, where $m \geq n \geq o$ satisfy one of the following conditions:
 - * $m, n, o \geq 2$ with the same parity,
 - * $m, n \geq 3, o = 1$, and $\gcd(m+1, n+1) \neq 1$,
 - * $m \geq 2, n = o = 1$,
 - * $m, n \geq 2, o = 0$ and $\gcd(m, n) \neq 1$,
 - * $m \geq 2, n = 2$, and $o = 0$, or

- * $m = 2$ and $n = 1$.
- $T_n \vee (K_m + K_o)$, where $m \geq o$ and n satisfy one of the following conditions:
 - * $m, n, o \geq 2$ with the same parity,
 - * $m, o \geq 2, n = 1$, and $\gcd(m+1, o+1) \neq 1$,
 - * $m, n \geq 2, o = 1$, and $\gcd(m+1, n-1) \neq 1$,
 - * $m \geq 1, n = o = 1$,
 - * $n \geq 1, m = o = 1$,
 - * $m, n \geq 3, o = 0$, and $\gcd(m, n) \neq 1$,
 - * $m \geq 2, n = 2, o = 0$, or
 - * $m = 2, n \geq 2, o = 0$.

Proof. It turns out from theorems 4.7 and 4.8. □

5. THE SET $\mathbf{Forb}(\Gamma_{\leq k})$.

The characterization of the γ -critical graphs with a given algebraic co-rank, $\mathbf{Forb}(\Gamma_{\leq k})$, is very important. For instance, we were able to characterize $\Gamma_{\leq k}$ for k equal to 1 and 2 because we got a finite set of γ -critical graphs with algebraic co-rank equal to $k+1$ (for k equal to 1 and 2), and after that we proved that all the graphs that do not contain a graph from this set as an induced subgraph has algebraic co-rank less or equal to k . In this section we give two infinite families of forbidden simple graphs. This will prove that $\mathbf{Forb}(\Gamma_{\leq k})$ is not empty for all $k \geq 0$. Moreover, we conjecture that $\mathbf{Forb}(\Gamma_{\leq k})$ is finite for all $k \geq 0$. To finish we present an example of a simple graph G with algebraic co-rank equal to 5 but with no 5-minor equal to 1. That is, the 1 can be obtained uniquely from a non trivial algebraic combination of 5-minors of $L(G, X)$.

We begin by proving that the path with $n+2$ vertices is γ -critical with algebraic co-rank equal to $n+1$.

Theorem 5.1. *If $n \geq 0$, then $P_{n+2} \in \mathbf{Forb}(\Gamma_{\leq n})$.*

Proof. It is not difficult to prove $\gamma(P_{n+2}) = n+1$, see corollary 4.10 of [3]. On the other hand, if $H = P_{n+2} \setminus v$ for some $v \in V(P_{n+2})$, then H is a disjoint union of at most two paths. Let $H = P_{n_1} + \dots + P_{n_s}$ with $1 \leq s \leq 2$ and $\sum_{i=1}^s n_i = n+1$, then by lemma 3.7 we get that

$$\gamma(H) = \sum_{i=1}^s \gamma(P_{n_i}) = \sum_{i=1}^s (n_i - 1) = \sum_{i=1}^s n_i - s = n+1 - s < n+1.$$

Therefore $P_{n+2} \in \mathbf{Forb}(\Gamma_{\leq n})$. □

Now, we present another infinite family of graph that are γ -critical. Let K_n be the complete graph with n vertices and M_k a matching of K_n with k edges. We begin by finding the critical group of $K_n \setminus M_k$.

Proposition 5.2. *If K_n be the complete graph with n vertices and M_k is a matching of k edges, then*

$$K(K_n \setminus M_k) \cong \begin{cases} \mathbb{Z}_n^{n-2k-2} \oplus \mathbb{Z}_{n(n-2)}^k & \text{if } n \geq 2k+2, \\ \mathbb{Z}_{n-2} \oplus \mathbb{Z}_{n(n-2)}^{k-1} & \text{if } n = 2k+1. \end{cases}$$

Proof. If $n = 2k + 1$ the result follows by [7, Theorem 1]. Therefore we can assume that $n \geq 2k + 2$. Given $\mathbf{a} \in \mathbb{Z}^k$, let $N_{k+1}(\mathbf{a})$ be the matrix given by

$$\begin{bmatrix} 1 & \mathbf{a} \\ \mathbf{0}^t & I_k \end{bmatrix}.$$

If $M_k = \{v_1 v_2, \dots, v_{2k-1} v_{2k}\}$, then

$$L(K_n \setminus M_k, v_n) = \begin{bmatrix} [(n-2)I_2 + J_2] \otimes I_k - J_{2k} & -J_{2k, n-2k-1} \\ -J_{n-2k-1, 2k} & nI_{n-2k-1} - J_{n-2k-1} \end{bmatrix},$$

where \otimes is the tensor product of matrices. Now, since $\det(N_{n-1}(\mathbf{a})) = 1$ for all \mathbf{a} , then

$$\begin{aligned} L(K_n \setminus M_k, v_n) &\sim N_{n-1}(\mathbf{1})^t N_{n-1}(\mathbf{1}) L(K_n \setminus M_k, v_n) N_{n-1}(-\mathbf{1}) \\ &= I_1 \oplus nI_{n-2k-2} \bigoplus_{i=1}^k \begin{bmatrix} n-1 & 1 \\ 1 & n-1 \end{bmatrix}. \end{aligned}$$

On the other hand

$$\begin{aligned} \begin{bmatrix} n-1 & 1 \\ 1 & n-1 \end{bmatrix} &\sim \begin{bmatrix} 0 & 1 \\ -1 & n-1 \end{bmatrix} \begin{bmatrix} n-1 & 1 \\ 1 & n-1 \end{bmatrix} \begin{bmatrix} 1 & -(n-1) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & n(n-2) \end{bmatrix}. \end{aligned}$$

Therefore $L(K_n \setminus M_k, v_n) \sim I_{k+1} \oplus nI_{n-2k-2} \oplus n(n-2)I_k$. □

Corollary 5.3. *If $n = 2k + 2$, then $K_n \setminus M_k \in \mathbf{Forb}(\Gamma_{\leq k})$.*

Proof. First, by proposition 5.2 we have that

$$\gamma(K_n \setminus M_k) \leq \begin{cases} k+1 & \text{if } n \geq 2k+2, \\ k & \text{if } n = 2k+1. \end{cases}$$

Now, let $n \geq 2k+2$, $M_k = \{v_1 v_2, \dots, v_{2k-1} v_{2k}\}$, and $M = L(K_n \setminus M_k, X)[\{1, \dots, 2k+1\}, \{2, \dots, 2k+2\}]$ be a square submatrix of generalized Laplacian matrix of $K_n \setminus M_k$. Then

$$M = \begin{bmatrix} 0 & -1 & & -1 & -1 \\ -1 & 0 & & -1 & -1 \\ & & \ddots & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

By [3, theorem 3.13], $\det(M) = \det(L(K_k, X_{K_k}))|_{\{x_1=0, \dots, x_{k-1}=0, x_k=-1\}} \stackrel{3.13}{=} -1$ and therefore, $\gamma(K_n \setminus M_k) = k+1$ for all $n \geq 2k+2$. Finally, if $n = 2k+2$ and $v \in V(K_n \setminus M_k)$, then $(K_n \setminus M_k) \setminus v$ is equal to $K_{n-1} \setminus M_k$ or $K_{n-1} \setminus M_{k-1}$. Thus, $\gamma((K_n \setminus M_k) \setminus v) \leq k$ and therefore $K_n \setminus M_k \in \mathbf{Forb}(\Gamma_{\leq k})$. □

This results proves that $\mathbf{Forb}(\Gamma_{\leq k})$ is not empty for all $k \geq 0$.

Corollary 5.4. *If $k \geq 0$, then $\mathbf{Forb}(\Gamma_{\leq k})$ is not empty.*

For $i \geq 3$, the set $\mathbf{Forb}(\Gamma_{\leq i})$ is more complex than $\mathbf{Forb}(\Gamma_{\leq 1})$ and $\mathbf{Forb}(\Gamma_{\leq 2})$. For instance, in [1] was proved that $\mathbf{Forb}(\Gamma_{\leq 3})$ has 49 graphs. Moreover, we conjecture that $\mathbf{Forb}(\Gamma_{\leq k})$ is finite.

Conjecture 5.5. *For all $k \in \mathbb{N}$ the set $\mathbf{Forb}(\Gamma_{\leq k})$ is finite.*

Until now, all the graphs that were presented has algebraic co-rank equal to k because its generalized Laplacian matrix has a k -minor equal to one. Next example shows a graph G with $\gamma_{\mathbb{Z}}(G) = 5$ having no a 5-minor equal to 1.

Example 5.6. Let G be the graph on figure 2 and $f_1 = \det(L(G, X)[\{1, 2, 3, 4, 5\}; \{2, 3, 5, 6, 7\}]) =$

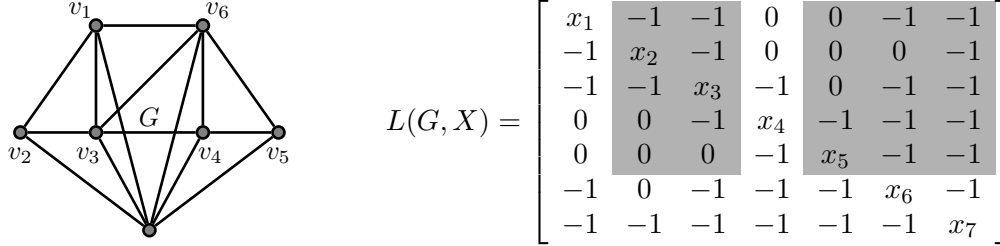


FIGURE 2. A graph G with seven vertices and its generalized Laplacian matrix.

$x_2 + x_5 + x_2x_5$, and $f_2 = \det(L(G, X)[\{1, 2, 3, 5, 6\}; \{2, 4, 5, 6, 7\}]) = -(1 + x_2 + x_5 + x_2x_5)$. Then $\langle f_1, f_2 \rangle = 1$ and therefore $\gamma_{\mathbb{Z}}(G) = 5$. However, it is not difficult to check that $L(G, X)$ has no 5-minor is equal to one.

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